There is the further advantage in that the integration interval from $s = 0$ to $s = 1$ is now fixed, thus simplifying the calculation of the coefficients in the linearized equations since these coefficients must be calculated at the end of each integration step.

The method given here can be generalized. Thus consider the case of a three-point boundary problem, involving say $t = 0$, $t = t_1$, $t = t_2$, where t_1 and t_2 are unknown. Such a situation can arise in rocket problems where it is necessary to switch from one mode of operation to another at time t_1. Then we write $t = as$, $0 \leqslant s \leqslant 1$, $t = a + b(s - 1)$, and $1 \leqslant s \leqslant 2$, with the boundary points given by $s = 0, 1, 2$, so that $t_1 = a$ and $t_2 = a + b$. It is necessary that t be a continuous, increasing function of s. Clearly the first of these conditions is satisfied. Also, since $0 \leqslant t_1 \leqslant t_2$, it follows that $a \geqslant 0$, $b \geqslant 0$, so that t is also an increasing function of s. Using the same method as before, and using an obvious convention concerning left-hand and right-hand derivatives at the end points of the intervals, the equation $\dot{x} = f$ becomes $x' = af$, $0 \leqslant s \leqslant 1$ and $x' = bf$, $1 \leqslant s \leqslant 2$; we treat a and b as state variables. The derivative x' now has a simple discontinuity at $s = 1$, but this does not affect the existence or uniqueness of the solution of the system and presents no difficulty in programing for a computer. Clearly this method can be extended to any number $n + 1$ of unknown boundary points $t_0, t_1, \ldots, t_n$ by taking the corresponding points as $s = 0, 1, \ldots, n$ and writing $t = a_1 + b_1 s$, $0 \leqslant s \leqslant 1$, $t = a_{r+1} + b_{r+1}(s - r)$, $r \leqslant s \leqslant r + 1$, and $r = 1, 2, \ldots, n - 1$. The continuity conditions are $a_{r+1} = a_r + b_r$ and $r = 1, 2, \ldots, n - 1$, and the values of t at the boundary points are given by $t_0 = a_1$, $t_r = a_r + b_r$, and $r = 1, 2, \ldots, n$.

Numerical Example

As an example we consider the system of equations $\ddot{x} = -x/r^3$ and $\ddot{y} = -y/r^3$ where $r^2 = x^2 + y^2$. The given boundary values are taken as

$$x(0) = 1.0 \qquad y(0) = 0.0$$
$$x(t_1) = 0.921061 \qquad y(t_1) = 0.389418$$
$$x(t_2) = 0.540303 \qquad y(t_2) = 0.841471$$

where both t_1 and t_2 are taken as unknown. [These values were in fact predetermined by taking the initial values $x(0) = 1.0$, $\dot{x}(0) = 0.0$, $y(0) = 0.0$, $\dot{y}(0) = 1.0$ together with $t_1 = 0.4$, $t_2 = 1.0$.] Applying the preceding method, we write $\dot{x} = u$, $\dot{y} = v$, and $t = as$ in $0 \leqslant s \leqslant 1$, $t = a + b(s - 1)$ in $1 \leqslant s \leqslant 2$, and so obtain the equations $x' = au$, $y' = av$, $u' = -ax/r^3$, and $v' = -ay/r^3$ with b replacing a in $1 \leqslant s \leqslant 2$. We now have six independent variables x, y, u, v, a, and b, and the linearization process is carried out with respect to these, in the manner explained in Ref. 1.

Starting functions were obtained for x and y by linear interpolation of the known values at $t = 0$, $t = t_2$ over the interval $0 \leqslant s \leqslant 2$. Values of t_1, t_2, and, hence, of a and b, were assumed, and corresponding values for u and v were assumed constant over the whole interval. For the seven forward integrations required for each iteration, initial vectors of the type (1, 0, 0, 0, 0, 0) were found satisfactory. A fourth-order Runge-Kutta scheme was used which, at the end of each iteration, required that the values of the variables at the midpoint of each integration step be calculated and be ready to provide the coefficients for the next iteration. These values were obtained by interpolation.

The following results were obtained with a step size in s of 0.1. Six digits were used in the calculations, and the results are shown to five decimal places. Table 1 shows the values of the indicated variables, obtained from the kth iteration. $x(0)$, $y(0)$ retain their given values and so are not shown. The first row ($k = 0$) contains the initial, assumed values, and the last row shows the true values.

In conclusion, it would appear that the method suggested affords a practical, efficient means of applying quasi-linearization to problems with undetermined end points.

Table 1 Values of variables from k th iteration.

k	$u(0)$	$v(0)$	t_1	t_2
0	−0.8	1.2	0.3	0.7
1	0.50802	0.42421	0.52782	1.28177
2	0.10588	0.86884	0.38348	1.05994
3	−0.00123	0.99448	0.40521	0.99043
4	−0.00016	1.00016	0.39990	0.99982
5	−0.00002	1.00000	0.39999	0.99999
...	0.0	1.0	0.4	1.0

Reference

[1] McGill, R. and Kenneth, P., "Solution of variational problems by means of a generalized Newton-Raphson operator," AIAA J. 2, 1761–1766 (1964).

Torsion of an Aeolotropic Cylinder Having a Spheroidal Inclusion on its Axis

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Introduction

RECENTLY, Bhowmick[1] has solved the problem of torsion of a circular cylinder of transversely isotropic material having a rigid spheroidal inclusion on its axis. Here we shall give the stresses due to an elastic inclusion in the form of a spheroid situated symmetrically on the axis of a large transversely isotropic circular cylinder under torsion.

Equations and Boundary Conditions

We take the origin at the center of the inclusion and the z axis along the axis of symmetry of the inclusion with the axis of the cylinder coinciding with the axis of figure of the inclusion. Let (r, θ, z) be the cylindrical coordinates of any point, and let (u, v, w), (u', v', w') be, respectively, the components of displacements outside the inclusion and internal to it in the increasing directions of r, θ, and z, respectively. The cylinder is assumed to be twisted about z axis. In the sequel, primed quantities will refer to the material of the inclusion.

When the material is under torsion, we assume

$$u = 0 = w \qquad v = v(r, z)$$
$$u' = 0 = w' \qquad v' = v'(r, z) \tag{1}$$

Then nonzero strain components are given by

$$e_{r\theta} = r\Phi_{,r} \qquad e_{\theta z} = r\Phi_{,z}$$
$$e_{r\theta}' = r\Phi_{,r}' \qquad e_{\theta z}' = r\Phi_{,z}' \tag{2}$$

where

$$\Phi = v/r \qquad \Phi' = v'/r \qquad \Phi_{,r} \equiv \partial\Phi/\partial r$$
$$\Phi_{,z} \equiv \partial\Phi/\partial z \qquad \text{etc.}$$

Nonzero stress components inside the inclusion are

$$p_{r\theta}' = Gr\Phi_{,r}' \qquad p_{\theta z}' = Gr\Phi_{,z}' \tag{3}$$

where G is the modulus of rigidity of the inclusion.

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The stress-strain relations in the case of a transversely isotropic material are given by

$$\left.\begin{aligned} p_{rr} &= c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz} \\ p_{\theta\theta} &= c_{12}e_{rr} + c_{11}e_{\theta\theta} + c_{13}e_{zz} \\ p_{zz} &= c_{13}e_{rr} + c_{13}e_{\theta\theta} + c_{33}e_{zz} \\ p_{\theta z} &= c_{44}e_{\theta z} \qquad p_{zr} = c_{44}e_{zr} \qquad p_{r\theta} = c_{66}e_{r\theta} \end{aligned}\right\} \quad (4)$$

where $c_{12} = c_{11} - 2c_{66}$.

The nonvanishing stress components outside the inclusion are therefore given by

$$p_{r\theta} = c_{66}r\Phi_{,r} \qquad p_{\theta z} = c_{44}r\Phi_{,z} \quad (5)$$

Body stress equations of equilibrium are satisfied if

$$(r^3\Phi_{,r})_{,r} + k^2(r^3\Phi_{,z})_{,z} = 0 \qquad (r^3\Phi_{,r}')_{,r} + (r^2\Phi_{,z}')_{,z} = 0 \quad (6)$$

where $k^2 \equiv c_{44}/c_{66}$.

Equations (6) indicate the existence of functions Ψ and Ψ' such that

$$\left.\begin{aligned} r^3\Phi_{,r} &= -k\Psi_{,z} \qquad r^3\Phi_{,z} = (1/k)\Psi_{,r} \\ r^3\Phi_{,r}' &= -\Psi_{,z}' \qquad r^3\Phi_{,z}' = \Psi_{,r}' \end{aligned}\right\} \quad (7)$$

Therefore

$$\left.\begin{aligned} p_{r\theta} &= c_{66}r\Phi_{,r} = -kc_{66}\cdot(1/r^2)\Psi_{,z} \\ p_{\theta z} &= c_{44}r\Phi_{,z} = (kc_{66}/r^2)\cdot\Psi_{,r} \\ p_{r\theta}' &= Gr\Phi_{,r}' = -G\cdot(1/r^2)\Psi_{,z}' \\ p_{\theta z}' &= Gr\Phi_{,z}' = (G/r^2)\Psi_{,r}' \end{aligned}\right\} \quad (8)$$

It follows from Eq. (8) that

$$\begin{aligned} \Psi_{,rr} - (3/r)\Psi_{,r} + k^2\Psi_{,zz} &= 0 \\ \Psi_{,rr}' - (3/r)\Psi_{,r}' + \Psi_{,zz}' &= 0 \end{aligned} \quad (9)$$

The equation to the surface S of the inclusion is

$$r^2/a^2 + z^2/b^2 = 1 \quad (10)$$

where $a > b$ in the case of an oblate spheroid and $a < b$ if the spheroid is prolate. The boundary conditions are as follows:

1) At a great distance from the inclusion, the effect of the inclusion being inappreciable,

$$\Psi \sim \tfrac{1}{4}\tau k r^4 \quad (11a)$$

τ being the twist.

2) On S,

$$\Phi = \Phi' \quad (11b)$$

3) Finally, the condition of continuity of stress across S gives

$$p_{\theta z}(dr/ds) - p_{r\theta}(dz/ds) = p_{\theta z}'(dr/ds) - p_{r\theta}'(dz/ds)$$

where s is measured along a meridian arc on S. With the help of Eq. (8), this gives

$$G(d\Psi'/ds) = kc_{66}(d\,\Psi/ds)$$

and hence, discarding an arbitrary constant, we write

$$G\Psi' = kc_{66}\Psi \quad (11c)$$

Solution of the Problem

Putting $z = kz'$, the equation for Ψ in Eq. (9) and the surface equation (10) become

$$\Psi_{,rr} - (3/r)\Psi_{,r} + \Psi_{,z'z'} = 0 \quad (12)$$

$$r^2/a^2 + z'^2/(b^2/k^2) = 1 \quad (13)$$

Three cases may arise accordingly as k is less than, or greater than, or equal to b/a.

***Case 1:* $k < b/a$**

We introduce the transformation of coordinates given by

$$z' + ir = c\cosh(\xi + i\eta) \qquad i = (-1)^{1/2} \quad (14)$$

Putting

$$c = (b^2/k^2 - a^2)^{1/2} \qquad \tanh\alpha = ak/b$$

we have $\xi = \alpha$ on the surface equation (13). At great distances from the inclusion $\xi = \alpha$, $\xi \to \infty$.

In (ξ, η) coordinates, Eq. (12) is

$$\Psi_{,\xi\xi} + \Psi_{,\eta\eta} - 3\coth\xi\Psi_{,\xi} - 3\cot\eta\Psi_{,\eta} = 0 \quad (15)$$

Now for large ξ, condition [Eq. (11a)] being

$$\Psi \sim \tfrac{1}{4}\tau k c^4 \sinh^4\xi \sin^4\eta$$

we take a solution of Eq. (15) in the form

$$\Psi = c^4(1 - \mu^2)^2(\nu^2 - 1)^2[(\tau k/4) + B_2Q_2''(\nu)] \quad (16)$$

where B_2 is a constant, $\mu = \cos\eta$, $\nu = \cosh\xi$, and Q_2 is the Legendre's function of degree two, and of the second kind, where $Q_2'(\nu) \equiv dQ_2(\nu)/d\nu$, $Q_2''(\nu) = d^2Q_2(\nu)/d\nu^2$.

Remembering what z' means, we can show by Eqs. (7) and (14) that

$$\Phi_{,\xi} = (1/r^3)\Psi_{,\eta} \qquad \Phi_{,\eta} = -(1/r^3)\Psi_{,\xi} \quad (17)$$

Hence

$$\Phi = 4\mu c[(\tau k\nu/4) + B_2Q_2'(\nu)] \quad (18)$$

As a solution of the differential equation for Ψ' in Eq. (9), we take

$$\Psi' = Ar^4 \quad (19)$$

where A is a constant. Hence

$$\Phi' = 4^{Az} = 4Akc\mu\nu \quad (20)$$

Now using Eqs. (11b, 18, and 20) and using Eqs. 11c, 16, and 19) we have

$$(\tau k\nu_0/4) + B_2Q_2'(\nu_0) = Ak\nu_0 \quad (21)$$

$$kc_{66}[(\tau_k/4) + B_2Q_2''(\nu_0)] = GA \text{ where } \nu_0 = \cosh\alpha$$

Solving Eqs. (21),

$$\left.\begin{aligned} A &= \frac{\tau k^2}{4}\cdot c_{66}\cdot\frac{Q_2'(\nu_0) - \nu_0Q_2''(\nu_0)}{GQ_2'(\nu_0) - k^2c_{66}\nu_0Q_2''(\nu_0)} \\ B_2 &= \frac{\tau k\nu_0}{4}\cdot\frac{k^2c_{66} - G}{GQ_2'(\nu_0) - k^2c_{66}\nu_0Q_2''(\nu_0)} \end{aligned}\right\} \quad (22)$$

***Case 2:* $k > b/a$**

Here we set

$$r + iz' = c\cosh(\xi + i\eta) \quad (23)$$

Putting

$$c = (a^2 - b^2/k^2)^{1/2} \qquad \tanh\alpha = b/ak$$

we have $\xi = \alpha$ on the surface equation (13). Equation (12) is now

$$\Psi_{,\xi\xi} + \Psi_{,\eta\eta} - 3\tanh\xi\Psi_{,\xi} + 3\tan\eta\Psi_{,\eta} = 0 \quad (24)$$

and corresponding to relations (17) we have now

$$\Phi_{,\xi} = -(1/r^3)\Psi_{,\eta} \qquad \Phi_{,\eta} = (1/r^3)\Psi_{,\xi} \quad (25)$$

It can be shown that Ψ's and Φ's, in this case, are given by

$$\Psi = r^4[(\tau k/4) + B_2 Q_2''(\nu)] \qquad \Psi' = Ar^4 \qquad (26)$$

$$\Phi = (4\mu c/i)[(\tau k\nu/4) + B_2 Q_2'(\nu)] \qquad \Phi' = 4Akc\mu\nu/i$$

where

$$\mu = \sin\eta \qquad \nu = i\sinh\xi$$

and A, B_2 are constants. They are given by Eq. (22) where ν_0 is now equal to $i\sinh\alpha$. The result obtained by Bhowmick[1] follows immediately from the preceding case when $G \to \infty$.

***Case 3:* $k = b/a$**

Here we set

$$z' + ir = \rho e^{i\varphi} \qquad (27)$$

so that, on the surface equation (13),

$$\rho = a \qquad (28)$$

and Eq. (12) transforms into

$$\Psi_{,\rho\rho} + (1/\rho^2)\Psi_{,\varphi\varphi} - (2/\rho)\Psi_{,\zeta} - 3(\cot\varphi/\rho^2)\Psi_{,\varphi} = 0 \qquad (29)$$

Now for large ρ, condition (11a) being

$$\Psi \sim (\tau k/4)\rho^4(1 - \mu^2)^2$$

where $\mu = \cos\varphi$, we take a solution of Eq. (29) in the form

$$\Psi = (1 - \mu^2)^2[(\tau k/4)\rho^4 + B_2/\rho] \qquad (30)$$

Since Φ and Ψ can be shown to satisfy the following relations,

$$r^3\Phi_{,\rho} = (1/\rho)\Psi_{,\varphi} \qquad r^3\Phi_{,\varphi} = -\rho\Psi_{,\rho} \qquad (31)$$

we therefore have

$$\Phi = \mu\rho(\tau k - B_2/\rho^5) \qquad (32)$$

We take Ψ′ and consequently Φ′ in the forms

$$\Psi' = Ar^4 = A\rho^4(1 - \mu^2)^2 \qquad \Phi' = 4Ak\mu\rho \qquad (33)$$

Now $\rho = a$, $\Phi = \Phi'$, and $G\Psi' = kc_{66}\Psi$. These conditions lead to the following values of A and B_2:

$$A = (5\tau k^2/4)\cdot[c_{66}/(G + 4k^2c_{66})]$$

$$B_2 = \tau k a^5\cdot(G - k^2c_{66})/(G + 4k^2c_{66}) \qquad (34)$$

Stresses

The important stress component on the surface of the inclusion is

$$p_{n\theta} = (G/r^2)\cdot(\partial\Psi'/\partial s) \qquad (35)$$

where s is measured along a meridian arc on the surface of the inclusion, and n is the normal to this surface. Therefore,

$$p_{n\theta} = 2AGa^2\sin 2\zeta/(a^2\cos^2\zeta + b^2\sin^2\zeta^{1/2}$$

where $\zeta = \eta$, or $\pi/2 - \eta$, or φ accordingly as k is less than, or greater than, or equal to b/a and the constant A is known in all the cases.

Reference

[1] Bhowmick, S. K., "Stress concentrations around a small rigid spheroidal inclusion on the axis of a transversely isotropic cylinder under torsion," AIAA J. 1, 1219–1220 (1963).

An Analysis of a Compressible, Turbulent Boundary Layer on a Chemically-Reacting Pyrolytic Boron Nitride Surface

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Introduction

THE developments over the past few years in the production of pyrolytic materials by decomposition of gases have created a growing interest in this type of material. Intensive investigations have been conducted on pyrolytic graphite, and in many respects pyrolytic boron nitride is similar to pyrolytic graphite.[1] The differences in the hot pressed and pyrolytic forms of boron nitride (BN) are much larger than the corresponding differences in the sintered and pyrolytic form of graphite. Hot pressed BN is limited primarily by the low melting temperature of the binder, B_2O_3, and the rapid reaction of B_2O_3 with water vapor.[2] The pyrolytic form has removed this limitation and extended the useful temperature range considerably.

There are very few published reports on the oxidation phenomena of pyrolytic BN. Some experimental data are available from the manufacturers. This data is in the form of measured weight loss at various temperatures at 1-atm pressure and with low-velocity air flow. The effects of oxide films and nitrogen release in protecting the surface from oxidation cannot be evaluated from the data. Powers[3] has published the results of oxidation tests on both boron nitride and a titanium diboride-boron nitride composition, but the data was not suited for an accurate determination of mass-loss rate. A theoretical study of the combustion phenomena with laminar flow has been performed by Bowen and Gorton.[4]

In this work the reactions of a pyrolytic boron nitride surface and air with a turbulent, compressible boundary layer are analyzed. The latest available thermochemical data on the boron species are incorporated into the mass-transfer calculations. The pressure and temperature corresponding to saturation by B_2O_3 (at low temperatures) and B (at high temperatures) are determined. A comparison is given of the heat and mass transfer occurring for a range of U_L/U_E from fully turbulent to laminar flow.

Analysis

The equations for determining the element mass fractions K at the wall in terms of mass-transfer parameter B' and freestream concentrations may be obtained by modifying the method given by Lees[5] for a "pure element" surface.[4] It is further assumed in the present analysis that all chemical reactions occur at the wall, and the products of reaction are distributed through the gas boundary layer by diffusion. The composition at the wall is assumed frozen at the chemical equilibrium concentration for a given wall temperature. The gas mixture is assumed to be binary insofar as diffusion is concerned. Under these assumptions it can be shown that

$$\bar{K}_{ie}/\bar{K}_{iw} = B' + 1 \qquad (1)$$

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